Measuring MRI noise

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Appendix B

Mathematical derivations

B.1 Fourier transform of frequency sweep signal

The known Fourier transforms are:

$$\begin{aligned}
\mathcal{F}\big\{\cos(\omega_0 t)\big\} &= \sqrt{\frac{\pi}{2}}\big(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\big) \\
\mathcal{F}\big\{\sin(\omega_0 t)\big\} &= \imath\sqrt{\frac{\pi}{2}}\big(\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\big)
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
\mathcal{F}\big\{\cos(\beta t^2)\big\} &= \frac{1}{\sqrt{4\beta}}\big(\cos\big[\frac{\omega^2}{4\beta}\big]+\sin\big[\frac{\omega^2}{4\beta}\big]\big) \\
\mathcal{F}\big\{\sin(\beta t^2)\big\} &= \frac{1}{\sqrt{4\beta}}\big(\cos\big[\frac{\omega^2}{4\beta}\big]-\sin\big[\frac{\omega^2}{4\beta}\big]\big).
\end{aligned} \tag{B.2}$$

Using the trigonometric expansion for formula 3.12 yields:

$$\begin{aligned}
&\mathcal{F}\big\{\cos\big((\omega_0+\beta t)t\big)\big\} = \\
&\mathcal{F}\big\{\cos(\omega_0 t+\beta t^2)\big\} = \\
&\mathcal{F}\big\{\cos(\omega_0 t)\cos(\beta t^2)-\sin(\omega_0 t)\sin(\beta t^2)\big\} = \\
&\mathcal{F}\big\{\cos(\omega_0 t)\cos(\beta t^2)\big\}-\mathcal{F}\big\{\sin(\omega_0 t)\sin(\beta t^2)\big\} = \\
&\mathcal{F}\big\{\cos(\omega_0 t)\big\}*\mathcal{F}\big\{\cos(\beta t^2)\big\}-\mathcal{F}\big\{\sin(\omega_0 t)\big\}*\mathcal{F}\big\{\sin(\beta t^2)\big\}.
\end{aligned} \tag{B.3}$$

Replacing the separate terms in equation B.3 with those from equations B.1 and B.2, and working out the convolutions, results in:

$$\begin{aligned}
&\mathcal{F}\big(\cos\big((\omega_0+\beta t)t\big)\big) = \\
&\frac{1}{4\sqrt{\beta}}\Big\{(1-\imath)\cos\left(\frac{(\omega-\omega_0)^2}{4\beta}\right)+(1+\imath)\sin\left(\frac{(\omega-\omega_0)^2}{4\beta}\right)+ \\
&\qquad(1+\imath)\cos\left(\frac{(\omega+\omega_0)^2}{4\beta}\right)+(1-\imath)\sin\left(\frac{(\omega+\omega_0)^2}{4\beta}\right)\Big\}.
\end{aligned} \tag{B.4}$$

Taking the Fourier transform of equation B.4 convolved with a rectangular window between $T_c - T$ and $T_c + T$ leads, according to Mathematica 5.0 (Wolfram Research, Inc.), to:

$$\begin{aligned}
&\int_{T_c-T}^{T_c+T} \cos\big((\omega_c + \beta t)t\big) e^{-\imath\omega t}\mathrm{d}t = \\
&\quad \frac{1}{\sqrt{\beta}}\Bigg\{ (\frac{1}{4} + \frac{\imath}{4})\sqrt{\frac{\pi}{2}}\Bigg(\imath\cos\left[\frac{(\omega-\omega_c)^2}{4\beta}\right] + \sin\left[\frac{(\omega-\omega_c)^2}{4\beta}\right]\Bigg)\times \\
&\quad \Bigg[\mathrm{erfi}\left[\frac{(\frac{1}{2}+\frac{\imath}{2})\big(2\beta(T_c - T) - (\omega-\omega_c)\big)}{\sqrt{2\beta}}\right] - \\
&\quad \mathrm{erfi}\left[\frac{(\frac{1}{2}+\frac{\imath}{2})\big(2\beta(T_c + T) - (\omega-\omega_c)\big)}{\sqrt{2\beta}}\right] + \\
&\quad \Bigg(\cos\big[\frac{\omega^2+\omega_c^2}{2\beta}\big] + \imath\sin\big[\frac{\omega^2+\omega_c^2}{2\beta}\big]\Bigg)\times \\
&\quad \Bigg(\mathrm{erf}\left[\frac{(\frac{1}{2}+\frac{\imath}{2})\big(2\beta(T_c - T) + (\omega+\omega_c)\big)}{\sqrt{2\beta}}\right] - \\
&\quad \mathrm{erf}\left[\frac{(\frac{1}{2}+\frac{\imath}{2})\big(2\beta(T_c + T) + (\omega+\omega_c)\big)}{\sqrt{2\beta}}\right]\Bigg)\Bigg]\Bigg\},
\end{aligned} \tag{B.5}$$

where ω_c is the frequency at T_c. This is graphically depicted in figure 4.2, panels A and B. The results of multiplying the time signal with a Kaiser-Bessel window are in panels C through F. These graphs are obtained with Digital Fourier Transforms in Matlab (The MathWorks, Inc.).

B.2 Noise cancellation

The addition of two sinusoidal signals in opposite phase may lead to cancellation. This principle is used in active noise cancellation (section 5.4.2). The remaining signal s is:

$$\begin{aligned}
s &= a_1\sin(\omega t) - (a_1 + \delta a)\sin(\omega t + \delta\phi) \\
&= -(a_1+\delta a)\big[\sin(\omega t)\cos(\delta\phi) + \cos(\omega t)\sin(\delta\phi)\big] + a_1\sin(\omega t).
\end{aligned} \tag{B.6}$$

With perfect phase matching ($\delta\phi = 0$), the remaining signal $s = -d\sin(\omega t)$, and for perfect amplitude matching ($\delta a = 0$),

$$s = -a_1\big[\sin(\omega t)\cos(\delta\phi) + \cos(\omega t)\sin(\delta\phi)\big] + a_1\sin(\omega t). \tag{B.7}$$

This can be approximated by

$$s = -a_1\delta\phi\cos(\omega t), \tag{B.8}$$

when $\delta\phi$ is small.